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Fault diagnosis for $t/(t+1)$-diagnosable system based on the PMC model①

Xie Min (谢 敏)[*], Liang Jiarong②[* **], Zhang Qian[* **], Chen Haiqiang[* **]
([*] School of Computer and Electronic Information, Guangxi University, Nanning 530004, P. R. China)
([**] Guangxi Key Laboratory of Multimedia Communications and Network Technology, Nanning 530004, P. R. China)

Abstract

In this paper, a $t/(t+1)$-diagnosable system is studied, which can locate a set S with $|S| \leqslant t+1$ containing all faulty units only if the system has at most t faulty units. On the basis of the characterization of the $t/(t+1)$-diagnosable system, a necessary and sufficient condition is presented to judge whether a system is $t/(t+1)$-diagnosable. Meanwhile, this paper exposes some new and important properties of the $t/(t+1)$-diagnosable system to present the $t/(t+1)$-diagnosability of some networks. Furthermore, the following results for the $t/(t+1)$-diagnosability of some special networks are obtained: a hypercube network of n-dimensions is $(3n-5)/(3n-4)$-diagnosable, a star network of n-dimensions is $(3n-5)/(3n-4)$-diagnosable ($n \geqslant 5$) and a 2D-mesh (3D-mesh) with n^2 (n^3) units is 8/9-diagnosable (11/12-diagnosable). This paper shows that in general, the $t/(t+1)$-diagnosability of a system is not only larger than its t/t-diagnosability, but also its classic diagnosability, specially the $t/(t+1)$-diagnosability of the hypercube network of n-dimensions is about 3 times as large as its classic t-diagnosability and about 1.5 times as large as its t/t-diagnosability.

Key words: t/t-diagnosable system, characterization of $t/(t+1)$-diagnosable system, fault diagnosis, n-dimensional hypercube networks

0 Introduction

As a result of the rapid development in digital technology, a multiprocessor computer system can become a system incorporating hundreds and thousands of processors (units). The number of processors in such a system is so large that it is difficult to avoid the phenomenon that some processors become faulty, especially after the system continues to work for a long time. To maintain reliability, the system should be designed to have the ability of identifying faulty processors that can be repaired or changed by additional ones. On the identification of the fault processors in the system, there are two approaches, one is called the logic-electric-level approach, the other is system-level approach. Because of the large number of processors in a system, the approach of the fault identification has been inclined to emphasize the system-level approach rather than the logic-electric-level approach[1]. In the models of system-level fault diagnosis, the model proposed by Preparata et al, called the PMC (preparata, Metze, and Chien) model, is known as the first model of system-level diagnosis[2,3]. For a system denoted by digraph $H(V,E)$, the PMC model assumes that $(a,b) \in E$ represents unit b is tested by unit a. $\mu(a,b)$ stands for the test result of unit a testing unit b. If a judges b to be fault-free, then $\mu(a,b)=0$. Otherwise, $\mu(a,b)=1$. If a is fault-free, then the result of a testing b is reliable. Otherwise, unreliable. The results about the PMC model can be found in Refs[4-12].

Under the PMC model, t-diagnosis[1] and t/t-diagnosis[13] are two important diagnosis strategies in the system-level diagnosis. If a system with at most t faulty units can correctly identify all units in it, then it is called a t-diagnosable system. And a system with at most t faulty units is called a t/t-diagnosable system if a set with the size of at most t can be located so that it incorporates all faulty units. There are some papers reporting the results about t/t-diagnosable system[8,14-17]. However, for a t-diagnosable (t/t-diagnosable) system, if the real number of faulty units $t' > t$, then the above two diagnosable strategies can do nothing for identifying the faulty units in the system. In other

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② To whom correspondence should be addressed. E-mail: 13977106752@163.com
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words, in designing a system, it should have a large diagnosability, which is the maximum number of faulty units the system can guarantee to identify. This gives us a strong motivation to study on the $t/(t+1)$-diagnosable system, whose diagnosability is larger than that of the t-diagnosable system and that of t/t-diagnosable system.

In next section a characterization of the $t/(t+1)$-diagnosable systems based on the PMC model will be proposed. In Section 2 some important properties on the $t/(t+1)$-diagnosable systems are presented. In Section 3 the diagnosability of some special networks are analyzed such as n-dimensional hypercube network and n-dimensional 2D(3D)-mesh network.

1 Characterization of the $t/(t+1)$- diagnosable systems

For the sake of convenience, let us introduce some necessary preparation knowledges. A diagnostic graph of a system or a network S with n units is usually denoted by an undirected graph $H(V,E)$, where V denotes the set of units in S, E is edge set, $(u_i, u_j) \in E$ implies u_i tests u_j. For $x \in V$, two sets are defined as follows: $\Gamma x = \{y \mid (x, y) \in E\}$ and $\Gamma^{-1}x = \{y \mid (y, x) \in E\}$. For $U \subseteq V$, $\Gamma U = \cup_{y \in U} \Gamma y - U$, $\Gamma^{-1}U = \cup_{y \in U} \Gamma^{-1}y - U$ are defined.

Remarks When $H(V,E)$ is an undirected graph, $\Gamma x = \Gamma^{-1}x$, $\Gamma U = \Gamma^{-1}U$. The relative terminologies and notations in this paper follow Ref. [18]. Throughout this paper, the four notations: unit, node, vertex and processor are not distinguished. At the same time, also other three notations: graph, system and network are not distinguished.

Let $H(V,E)$ stand for a network graph, $V(H)$ stand for the unit set in H. A connected subgraph L in H is called a connected component if there doesn't exist an edge $(x, y) \in E$ such that $x \in V(L)$ and $y \in V - V(L)$. $C_{comp}(H) = \{C_i \mid 1 \leqslant i \leqslant k\}$ is used to stand for the set of all connected component in H. It is obvious that for $C_i, C_j \in C_{comp}(H) (i \neq j)$, $V(C_i) \cap V(C_j) = \varnothing$, $V(C_1) \cup V(C_2) \cup \cdots V(C_k) = V$.

Especially, if H is connected, $C_{comp}(H) = H$.

Let $L \subset V$, $H_{ind}(L) = (L, E')$ with $E' = \{(x, y) \in E \mid x, y \in L\}$ is said to be the induced subgraph of L in H. Let $Card_k(C_{comp}(H)) = \{C_i \mid C_i \in C_{comp}(H), \mid C_i \mid = k\}$. In order to explain the two terminologies $C_{comp}(H)$ and $Card_k(C_{comp}(H))$, let us consider an example shown in Fig. 1. Fig. 1 is a graph of 10-node, denoted by $H(V, E)$, which is easily obtained that $C_{comp}(H) = \{C_1, C_2, C_3\}$, where $C_1 = H_{ind}(v_1, v_2)$, $C_2 = H_{ind}(v_3, v_4, v_5)$ and $C_3 = H_{ind}(v_6, v_7, v_8, v_9, v_{10})$. $Card_2(C_{comp}(H)) = \{C_1\}$, $Card_3(C_{comp}(H)) = \{C_2\}$, $Card_5(C_{comp}(H)) = \{C_3\}$ and $Card_4(C_{comp}(H)) = \varnothing$.

Definition 1 System S with at most t faulty units is said to be $t/(t+1)$-diagnosable if a set of size at most $t+1$ can always be determined so that it incorporates all faulty units.

The concept of allowable fault set (AFS) is important in fault diagnosis, the definition of AFS is as follows[14].

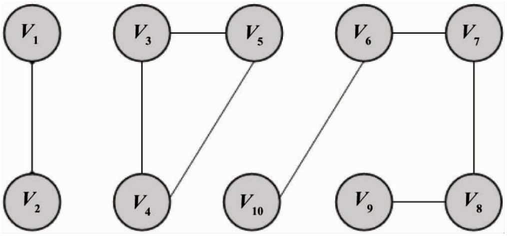

Fig. 1 A graph of 10-node

Definition 2 Let $H(V, E)$ be a graph, μ a syndrome, Y a subset of V. For syndrome μ if the following conditions are satisfied, Y is said to be an *AFS* of the system.

i) For $(a, b) \in E$, if $a, b \in V - Y$, then $\mu(a, b) = 0$, and

ii) For $(a, b) \in E$, if $a \in V - Y$ and $b \in Y$, then $\mu(a, b) = 1$.

Lemma 1 [14] Let $H(V, E)$ be a graph, μ a syndrome, $P \subset V$ and $Q \subset V$. If both P and Q are *AFSs* for μ, then $P \cup Q$ is also an *AFS*.

For convenience, for any subsets $A, B \subseteq V$, let $(A - B) \cup (B - A) = A\Delta B$.

Lemma 2 [9] A system $H(V, E)$ is t/t-diagnosable if and only if any two subsets A, $B \subseteq V$ with $\mid A \mid \leqslant t+1$, $\mid B \mid \leqslant t+1$ and $A \neq B$, there exists an edge from $V - A - B$ to $A\Delta B$.

Theorem 1 A system given by $H(V, E)$ is t/t-diagnosable if for any two subsets $A, B \subseteq V$ with $\mid A \mid \leqslant t+1$, $\mid B \mid \leqslant t+1$ and $A \neq B$, there exists an edge from V-A-B to $A\Delta B$.

Proof According to Lemma 2, $H(V,E)$ is $(t+1)/(t+1)$-diagnosable, which implies that $H(V, E)$ is $t/(t+1)$-diagnosable.

Theorem 2 If a system given by $H(V, E)$ is $t/(t+1)$-diagnosable, then for any two subsets $A, B \subseteq V$ with $\mid A \mid \leqslant t$, $\mid B \mid \leqslant t$ and $A \neq B$, at least one condition holds in the following conditions.

i) There exists an edge from $V - A - B$ to $A\Delta B$.

ii) $\mid A \cup B \mid \leqslant t+1$.

Proof Suppose, on the contrary, that $A, B \subseteq V$

with $|A| \leqslant t$, $|B| \leqslant t$ and $A \neq B$, such that the following two conditions are satisfied:

i) $\Gamma^{-1}(A \Delta B) \subseteq A \cap B$.

ii) $|A \cup B| \leqslant t + 2$.

Assume that the units of A are exactly all faulty units in the system. Consider the following syndrome μ for each pair of units a, $b \in E$ so that $(a, b) \in E$:

i) If a, $b \in V - A$, then $\mu(a, b) = 0$.

ii) If $a \in V - A$ and $b \in A$, then $\mu(a, b) = 1$.

iii) The test result from A to B is 1 and the test result from B to A is 1.

For the above syndrome, A and B are all allowable fault sets. Therefore, all the fault sets in a set of size at most $t + 1$ can't be isolated owing to $|A \cup B| \leqslant t + 2$.

Theorem 2 presents a necessary condition on $t/(t+1)$-diagnosable system, in the next section, some significant properties and corollaries of $t/(t+1)$-diagnosable system will be presented.

2 Properties of the $t/(t+1)$-diagnosable system

For a system $H(V, E)$ with n units, it is concluded that if it is t-diagnosable system, then $n \geqslant 2t + 1$[1]. It is easily seen that if $H(V, E)$ is t-diagnosable system, then it is also t/t-diagnosable, furthermore, it is also $t/(t+1)$-diagnosable. For the sake of convenience, $n \geqslant 2t + 1$ is assumed in the following argument. Before presenting the characterization of $t/(t+1)$-diagnosable system, some useful properties of the $t/(t+1)$-diagnosable system is summarized as follows:

Property 1 If $H(V, E)$ is t/t-diagnosable, then it is also $t/(t+1)$- diagnosable.

Lemma 3 Let $H(V, E)$ be a network system, $\lambda = \min\{|(\Gamma^{-1}v_1) \cup (\Gamma^{-1}v_3) \cup (\Gamma^{-1}v_2)|\}$ where v_1, v_2, $v_3 \in V$ and $(v_i, v_j) \notin E (1 \leqslant i \leqslant j \leqslant 3)$, then the system is not $(\lambda+1)/(\lambda+2)$- diagnosable.

Proof Without loss of generality, let three units v_1, v_2, $v_3 \in V$ such that $|(\Gamma^{-1}v_1) \cup (\Gamma^{-1}v_3) \cup (\Gamma^{-1}v_2)| = \lambda$. Let $L = (\Gamma^{-1}v_1) \cup (\Gamma^{-1}v_3) \cup (\Gamma^{-1}v_2)$. Suppose that $F = L \cup \{v_1\} \subset V$ is the set of faulty units. Then $|F| = \lambda + 1$. Consider a syndrome μ as follows for each pair of units $a, b \in E$ (shown in Fig. 2):

i) If $a, b \in V - F - \{v_2, v_3\}$, then $\mu(a,b) = 0$.

ii) If $a \in V - F$ and $b \in F - \{v_2, v_3\}$, then $\mu(a, b) = 1$.

iii) If $a \in \{v_1, v_2, v_3\}$ and $b \in F$, then $\mu(a,b) = 1$.

iv) If $a \in \{v_1, v_2, v_3\}$ and $b \in V - F$, then $\mu(a, b) = 0$.

v) Other situations $\mu(a,b)$ is 0 or 1.

For the above syndrome μ, $L \cup \{v_i\} (1 \leqslant i \leqslant 3)$ are all allowable fault sets. Since $|\cup_{i=1}^{3}[L \cup \{v_i\}]| = \lambda + 3 > \lambda + 2$, it is concluded that the system is not $(\lambda+1)/(\lambda+2)$-diagnosable.

Lemma 4 For a system given by $H(V,E)$ with n units, if the system is $t/(t+1)$-diagnosable, then for $X \subset V$ with $|X| = 2(t-r)$, $|\Gamma^{-1}X| > r$, where $r \in [0, t-1)$ is an integer.

Proof Assume that there exist an integer $r \in [0, t-1)$ and a subset $X \subset V$ with $|X| = 2(t-r)$, $|\Gamma^{-1}X| = s \leqslant r$. Decompose X into the two subsets X_1, X_2 such that $X = X_1 \cup X_2$ with $|X_1| = |X_2| = t - r$. Since $|V| - |X \cup \Gamma^{-1}X| = n - 2t + 2r - s \geqslant 1 + r + r - s$, one can choose a set $X_0 \subset V - (X \cup \Gamma^{-1}X)$ with $X_0 = r - s$. Let $Y_1 = \Gamma^{-1}X \cup X_1 \cup X_0$, $Y_2 = \Gamma^{-1}X \cup X_2 \cup X_0$, and $R = V - (Y_1 \cup Y_2)$. Note that $|Y_1| = |Y_2| = t$ and there are no edges from R to $(X_1 \cup X_2)$. Construct the following syndrome μ for each pair of units $a, b \in E$ such that $(a, b) \in E$ (shown in Fig. 3):

i) If $a, b \in R$, then $\mu(a,b) = 0$.

ii) If $a \in R$ and $b \in V - R$, then $\mu(a, b) = 1$.

iii) The other possible test results are arbitrary.

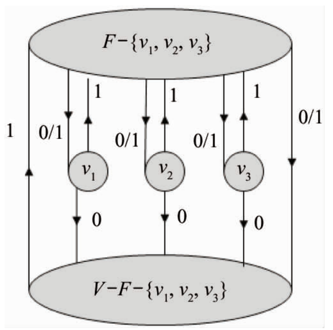

Fig. 2 A syndrome μ of Lemma 3

For above syndrome μ, Y_1 and Y_2 are all allowable fault sets. But $|Y_1 \cup Y_2| \geqslant t + 2$. Therefore, the system is not $t/(t+1)$-diagnosable.

Theorem 3 For a $t/(t+1)$-diagnosable system $H(V, E)$ and a syndrome σ, if both X and Y are $AFSs$, then at least one of following conditions holds:

i) $|X - X \cap Y| \leqslant 1$.

ii) $|Y - X \cap Y| \leqslant 1$.

Proof Let X and Y be two $AFSs$ with $|X - X \cap Y| \geqslant 2$ for syndrome σ, a contradiction will be derived. Without loss of generality, let $|X| = a$ and $|Y| = b$ with $t \geqslant a \geqslant b$. Since both of X and Y are $AFSs$, there exists no edge (u, v) such that $u \in V - X - Y$ and $v \in (X - Y) \cup (Y - X)$. Choose a set of units $Z \subset V - X - Y$ with $|Z| = t - a$.

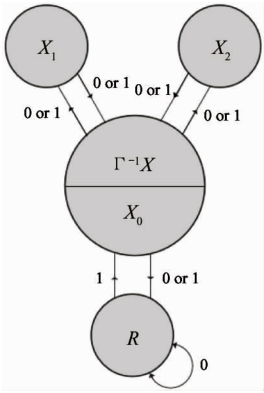

Fig. 3 A syndrome μ of Lemma 4

Let $X_z = X \cup Z$ and $Y_z = Y \cup Z$. Construct a syndrome σ_1 as follows: for two units $p, q \in V$ that $(p, q) \in E$ (shown in Fig. 4).

If $p, q \in V - X_z - Y_z$, then $\sigma_1(p, q) = \sigma(p, q) = 0$.

If $p \in V - X_z - Y_z$ and $q \in X_z \cap Y_z$, then $\sigma_1(p, q) = 1$.

The other possible situations $\sigma_1(p, q) = \sigma(p, q)$.

Since there exists no edge from $V - X - Y$ to $(X - Y) \cup (Y - X)$, therefore, there also exists no edge from $V - X_z - Y_z$ to $(X_z - Y_z) \cup (Y_z - X_z)$. It is obvious that for syndrome σ_1, both X_z and Y_z are all the allowable fault sets with $|X_z|$, $|Y_z| \leqslant t$. Note that $|X - X \cap Y| \geqslant 2$ and $|Y - X \cap Y| \geqslant 2$, which implies that $|X_z \cup Y_z| \geqslant t - a + a + 2 = t + 2$. Therefore, the system is not $t/(t+1)$-diagnosable system, a contradiction.

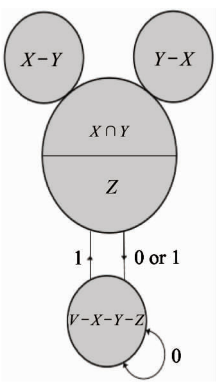

Fig. 4 A syndrome σ_1 of Theorem 3

Theorem 4 For a $t/(t+1)$-diagnosable system $H(V, E)$, a syndrome σ, suppose that both X and Y are *AFSs* with $X \nsubseteq Y$ and $Y \nsubseteq X$, $Z \subseteq V$, if $Z \not\subset X \cup Y$, then Z cannot be an *AFS* for syndrome σ.

Proof On the contrary, suppose that Z is an *AFS* for syndrome σ. Since $X \not\subset Y$, there exists a unit v that $v \in X$ and $v \notin Y$. By the assumption of both X and Y are *AFSs* and Lemma 1, we have $X \cup Y$ is also an *AFS*. Note that Theorem 3 and the assumption $Z \not\subset X \cup Y$, there exists a unit u that $u \in Z$ and $u \notin X \cup Y$. Hence, $|X \cup Z - Y| \geqslant 2$. On the other hand, since both $X \cup Z$ and Y are *AFSs* for syndrome σ, $|X \cup Z - Y| \leqslant 1$, a contradiction.

3 Applications

In this section, the t/t_1-diagnosability of the following three regular networks: n-dimensional hypercube network (Q_n), n-dimensional star network (S_n), and $2D$ ($3D$)-mesh will be discussed. For an undirected graph $H(V, E)$, subset $X \subset V$ and unit $v \in V$, let $\Gamma X = \Gamma X^{-1} = N(X)$ and $\Gamma v = \Gamma v^{-1} = N(v)$ where $X \subset V$, $v \in V$.

Lemma 5[19] Suppose that $H(V, E)$ is an undirected graph with n units. Let X be the faulty set in $H(V, E)$ with $|X| \leqslant t$, μ a syndrome produced by X, $Y \subset V$ with $|Y| \geqslant t+1$. If $H_{ind}(Y)$ is a connected subgraph and for any $(x, y) \in E$ such as $x, y \in Y$, $\sigma(x, y) = 0$, then $X \cap Y = \varnothing$, in other words each unit in Y is fault-free.

3.1 The $t/(t+1)$-diagnosability of n-dimensional hypercube network

Lemma 6 Suppose that $H(V, E)$ is the diagnostic graph of $Q_n (n \geqslant 4)$, $S = \{u, v, w\}$ is a subset of V in which u, v, w has one common neighbor. Then $|N(S)| = 3n - 5$.

Proof For unit $a \in V$, $add(a)$ is used to represent its address. Let unit x be the common neighbor of u, v, w. Without loss of generality, it is assumed that $add(u) = a_1 a_2 \cdots a_{n-3}100$, $add(v) = a_1 a_2 \cdots a_{n-3}010$, $add(w) = a_1 a_2 \cdots a_{n-3}001$, $add(x) = a_1 a_2 \cdots a_{n-3}000$. Note that any two units in S, say u and v, have exactly two common neighbors (one is x, the address of another one, say y, is $add(y) = a_1 a_2 \cdots a_{n-3}110$). At the same time, it is easily seen that each one of u, v, w has $n-3$ private neighbors. Therefore, $|N(S)| = 3(n-3) + 3 + 1 = 3n - 5$.

Lemma 7[20] Suppose that $H(V, E)$ is the diagnostic graph of Q_n, $L \subset V$ with $|L| = r$, $1 \leqslant r \leqslant n+1$, then $|N(L)| \geqslant rn - r(r+1)/2 + 1$.

Lemma 8[21] Suppose that $H(V, E)$ is the diagnostic graph of Q_n, $L \subset V$ with $|L| = r$. If $H_{ind}(V - L)$ is disconnected. Then following conditions hold:

i) If $n \leqslant r \leqslant 2n(r-1) - 1$, then $C_{comp}(H_{ind}(V - L)) = \{C_1, C_2\}$ with $|V(C_1)| = 1$ and $|V(C_2)| = 2^n - r - 1$.

ii) If $C_{comp}(H_{ind}(V - S)) = \{C_1, C_2, \cdots, C_k\}$ with

$|V(C_1)| \geq 2$ and $|V(C_2)| \geq 2$, then $|S| \geq 2(n-1)$.

Lemma 9 For 4-dimensional hypercube network Q_4 denoted by $H(V, E)$, let S be a set of units $S \subset V$, if there exists at least two C_i, $C_j \in C_{comp}(H_{ind}(V-S))$ with $|V(C_i)|$, $|V(C_j)| \geq 3$, then $|S| \geq 7$.

Proof According to Lemma 8, $|S| \geq 6$ is got. Next, $|S| \neq 6$ is shown. Assume that $|S| = 6$, then $|V-S| = 10$. Now the following cases are considered:

Case 1 $|V(C_i)| = 3$.

According to Lemma 6, $|N(V(C_i))| \geq 3n-5 = 7$, which is a contradiction to $N(V(C_i)) \subseteq S$. Therefore, no such subgraph $C_i \in C_{comp}(H_{ind}(V-S))$ with $|V(C_i)| = 3$ exists.

Case 2 $|V(C_i)| = 4$.

Similarly, according to Lemma 7, $|N(V(C_i))| \geq 4n-9 = 7$ is obtained, which is a contradiction to $N(V(C_i)) \subseteq S$. Therefore, no such subgraph $C_i \in C_{comp}(H_{ind}(V-S))$ with $|V(C_i)| = 4$ exists.

Case 3 $|V(C_i)| = 5$.

Q_4 can be divided into two 3-dimensional hypercube networks denoted by Q_3^L and Q_3^R. Consider the following cases:

Case 3.1 5 units are all in Q_3^L.

According to Lemma 7, $|N(V(C_i))| \geq 8$ is got, a contradiction.

Case 3.2 4 units are all in Q_3^L.

According to Lemma 7, $|N(V(C_i))| \geq 7$ is got, a contradiction.

Case 3.3 3 units are all in Q_3^L.

According to Lemma 7, $|N(V(C_i))| \geq 8$ is got, a contradiction.

In summary, no such subgraph $C_i \in C_{comp}(H_{ind}(V-S))$ with $|V(C_i)| = 3$ (4 or 5) exists that the result holds. Therefore, it can be seen the result holds only when $|S| \geq 7$.

Lemma 10 Suppose that $H(V, E)$ is the diagnostic graph of Q_n, $S \subset V$ with $2(n-1) \leq |S| \leq 3n-6$, $H_{ind}(V-S)$ is disconnected, $C_{comp}(H_{ind}(V-S)) = \{C_1, C_2, \cdots, C_k\}$. Then the following conditions hold:

i) $\sum_{k=1}^{2} k\,|Card_k(C_{comp}(H_{ind}(V-S)))| \leq 2$.

ii) There exists only one $C_i \in C_{comp}(H_{ind}(V-S))$ with $|V(C_i)| \geq 3$.

Proof If condition i) is not satisfied, then at least one of following cases must be satisfied:

Case 4 $|Card_1(C_{comp}(H_{ind}(V-S)))| \geq 3$.

There exists at least three units $u, v, w \in V-S$ such that $N(u, v, w) \subseteq S$. According to Lemma 6, $|N(u, v, w)| \geq 3n-5$, therefore, $N(u, v, w) \nsubseteq S$ which is a contradiction to the hypothesis.

Case 5 $|Card_2(C_{comp}(H_{ind}(V-S)))| \geq 2$.

There exists two pair adjacent units $v_1, v_2, v_3, v_4 \in V-S$ such that $N(\{v_1, v_2, v_3, v_4\}) \subseteq S$. According to Lemma 7, $|N(\{v_1, v_2, v_3, v_4\})| \geq 4n-9$, therefore, $N(\{v_1, v_2, v_3, v_4\}) \nsubseteq S$, which is a contradiction to the hypothesis.

Case 6 $|Card_1(C_{comp}(H_{ind}(V-S)))| \geq 1$ and $|Card_2(C_{comp}(H_{ind}(V-S)))| \geq 1$.

The argument here is similar to that of Case 4. According to the above three cases, condition i) must be satisfied.

The proof of condition ii): By induction on n, $|S| \geq 3n-5$. For $n = 4$, assume that $2(n-1) \leq |S| \leq 3n-6$, which implies $|S| = 6$. According to Lemma 9, it follows that if there exists at least two C_1, $C_2 \in C_{comp}(H_{ind}(V-S))$ is the two connected subgraph with $|V(C_1)|$, $|V(C_2)| \geq 3$, then $|S| \geq 7$, a contradiction. Hence, $n = 4$, the result is true. Assume that the result is true $n-1$ where $n-1 \geq 4$. For n, let C_1, $C_2 \in C_{comp}(H_{ind}(V-S))$ be two connected subgraph with $|V(C_1)| \geq 3$ and $|V(C_2)| \geq 3$. And dividing an n-dimensional hypercube into two copies Q_{n-1}^L and Q_{n-1}^R. Let $S = S_L \cup S_R$ where $S_L \in V(Q_{n-1}^L)$ and $S_R \in V(Q_{n-1}^R)$. Now the following cases are considered.

Case 7 $V(C_1) \subseteq V(Q_{n-1}^L)$ and $V(C_2) \subseteq V(Q_{n-1}^L)$.

According to the assumption, we have $|S_L| \geq 3(n-1)-5$. Since $V(C_1)(V(C_2))$ and $V((Q_{n-1}^R))$ is disconnected, $|S_R| \geq |V(C_1)| + |V(C_2)| \geq 6$. Then $|S| = |S_L| + |S_R| \geq 3(n-1)-5+6 = 3n-2 \geq 3n-5$, a contradiction.

Case 8 $V(C_1) \subseteq V(Q_{n-1}^L)$ and $V(C_2) \subseteq V(Q_{n-1}^R)$.

$|S_L| \geq |V(C_2)|$ and $|S_R| \geq |V(C_1)|$. If $|S_L| \leq 2(n-2)-1$, according to Lemma 7, $|V(C_1)| \geq 2^{n-1}-1-[2(n-2)-1] = 2^{n-1}-2n+4$. Then $|S_R| \geq 2^{n-1}-2n+4$ and when $n \geq 5$, $|S_R| \geq 3n-5$. If $|S_L| \geq 2(n-2)$, similarly, $|S_R| \geq 2(n-2)$, then $|S| \geq 4n-8 > 3n-5 (n \geq 5)$, a contradiction.

Case 9 $V(C_1) \subseteq V(Q_{n-1}^L) \cup V(Q_{n-1}^R)$ and $V(C_2) \subseteq V(Q_{n-1}^L)$.

Let $V(C_1) = A \cup B$ where $A \subseteq V(Q_{n-1}^L)$ and $B \subseteq V(Q_{n-1}^R)$. Now the following cases are considered.

Case 9.1 $|A| = 1$ and $|B| \geq 2$.

$|S_L| \geq n-1$ and $|S_L| \geq |V(C_2)| + |B| - |A|$. If $|B| = 2$, then $|S_R| \geq 2(n-2)$. Therefore, $|S| \geq (n-1) + 2(n-2) = 3n-5$. If $|B| \geq 3$, then $|S_R| \geq 3(n-1)-5$. Therefore, $|S| \geq (n-1) + 3(n-1) - 5 = 4n-9 \geq 3n-5 (n \geq 5)$, a contradiction.

Case 9.2 $|A| = 2$ and $|B| \geq 1$.

$|S_L| \geq 2(n-2)$ and $|S_R| \geq (n-1)$. Therefore, $|S| \geq (n-1) + 2(n-2) = 3n-5$, a contradiction.

Case 9.3 $|A| \geqslant 3$ and $|B| \geqslant 1$.

If $|B| \geqslant 3$, then $|S_R| \geqslant 3(n-1)-5$ and $|S_L| \geqslant |V(C_2)|$. Therefore, $|S| \geqslant 3(n-1)-5+3 = 3n-5$.

If $|B| = 2$, then $|S_R| \geqslant 2(n-2)$ and $|S_R| \geqslant |A| - |B|$. If $|S_L| < n-1$, then $|A| = 2^{n-1} - |S_L|$. Therefore, $|S| \geqslant 3n-5$. If $|S_L| \geqslant n-1$, then $|S| \geqslant (n-1)+2(n-2) = 3n-5$.

If $|B| = 1$, then $|S_R| \geqslant n-1$. If $n-1 \leqslant |S_L| < 2(n-2)-1$, according to Lemma 7, $|A| \geqslant 2^{n-1}-1-(2(n-2)-1) = 2^{n-1}-2n+4$. Then $|S| \geqslant (n-1)+2^{n-1}-2n+4-1 = 2^{n-1}-n+3 \geqslant 3n-5 (n \geqslant 5)$. It is easily seen that if $|S_L| < n-1$, $|S| \geqslant 3n-5$, a contradiction.

Case 10 $V(C_1) \subseteq V(Q^L_{n-1}) \cup V(Q^R_{n-1})$ and $V(C_2) \subseteq V(Q^L_{n-1}) \cup V(Q^R_{n-1})$.

Let $V(C_1) = A \cup B$ where $A \subseteq V(Q^L_{n-1})$ and $B \subseteq V(Q^R_{n-1})$. And let $V(C_2) \subseteq X \cup Y$ where $X \subseteq V(Q^L_{n-1})$ and $Y \subseteq V(Q^R_{n-1})$. Now see the following cases:

Case 10.1 $|A| + |X| = 2$.

$|S_L| \geqslant 2n-3$. Since $|B| \geqslant 2$ and $|Y| \geqslant 2$, $|S_R| \geqslant 2(n-2)$. and $|S| \geqslant 4n-7 \geqslant 3n-5 (n \geqslant 5)$, a contradiction.

Case 10.2 $|A| + |X| = 3$.

$|S_L| \geqslant 3(n-1)-5$ and $|S_R| \geqslant n-1$. $|S| \geqslant 4n-9 \geqslant 3n-5 (n \geqslant 5)$, a contradiction.

Case 10.3 $|A| + |X| \geqslant 4$.

If $n-1 \leqslant |S_L| \leqslant 2(n-2)-1$, then $|A| = 2^{n-1} - |S_L| - 1$ or $|X| = 2^{n-1} - |S_L| - 1$. Without loss of generality, let $|A| = 2^{n-1} - |S_L| - 1$. Then $|X| = 1$ and $|Y| \geqslant 2$. Therefore, $|Y| \leqslant |S_L| + 1 \leqslant 2(n-2)$. And if $|B| \geqslant 2$, then $|S_R| \geqslant 2(n-2)$. Otherwise, $|S_R| \geqslant |A| - |B|$. It is obviously that $|S| \geqslant 3n-5$, a contradiction.

If $|S_L| \geqslant 2(n-2)$, it is claimed that $|S_R| \geqslant n-1$. Otherwise, $H_{ind}(V(Q^R_{n-1}) - S_R, H)$ is connected. Then $|S| \geqslant 2(n-2)+(n-1) = 3n-5$, a contradiction.

Lemma 11 Suppose that $H(V, E)$ is the diagnostic graph of Q_n, $S \subset V$ with $|S| = 3n-5$. Suppose that $H_{ind}(V-S)$ is disconnected and let $C_{comp}(H_{ind}(V-S)) = \{C_1, C_2, \cdots, C_k\}$. Then following conditions hold:

i) $\sum_{k=1}^{3} k\,|Card_k(H_{ind}(V-S))| \leqslant 3$.

ii) There exists only one $C_i \in C_{comp}(H_{ind}(V-S))$ with $|V(C_i)| \geqslant 4$.

Proof a similar proof of Lemma 10 can be used to prove that the result is true.

Next, the $t/(t+1)$-diagnosability of n-dimensional hypercube network will be discussed.

Theorem 5 $Q_n (n \geqslant 5)$ is not $(3n-4)/(3n-3)$-diagnosable.

Proof According to Lemma 3 and Lemma 5, it is easily seen that the n-dimensional hypercube is not $(3n-4)/(3n-3)$-diagnosable.

Theorem 6 An n-dimensional hypercube $Q_n (n \geqslant 5)$, denoted by $H(V, E)$, is $(3n-5)/(3n-4)$-diagnosable.

Proof Suppose that $F \subseteq V$ with $|F| \leqslant 3n-5$ is the set of faulty units in Q_n.

Case 11 $|F| \leqslant n-1$.

It is obviously that $H_{ind}(V-F, H)$ is connected. According to Lemma 5, all nodes in $V-F$ can be correctly diagnosed to be fault-free.

Case 12 $n \leqslant |F| \leqslant 2(n-1)-1$.

When $H_{ind}(V-F, H)$ is connected, all units in $V-F$ can be diagnosed correctly to be fault-free. Contrarily, when $H_{ind}(V-F, H)$ is disconnected, according to Lemma 8, $C_{comp}(H_{ind}(V-F)) = \{C_1, C_2\}$ with $|C_1| = 1$ and $|C_2| = 2^n - 1 - |F|$. Since $|C_2| > |F|$, $V(C_2)$ can be identified correctly as fault-free. Furthermore, $V - V(C_2)$ can be located that it contains all faulty units in H.

Case 13 $2(n-1) \leqslant |F| \leqslant 3n-6$.

When $H_{ind}(V-F, H)$ is connected, all units in $V-F$ can be diagnosed correctly to be fault-free. Contrarily, when $H_{ind}(V-F, H)$ is disconnected, according to Lemma 10, $C_{comp}(H_{ind}(V-F)) = \{C_1, C_2, C_3\}$ ($V(C_3)$ may be empty). Since $|C_2| > |F|$ and $|V(C_2)| + |V(C_3)| \leqslant 2$, $V(C_1)$ can be identified correctly as fault-free. Furthermore, $V - V(C_1)$ can be located that it contains all faulty units in H.

Case 14 $|F| = 3n-5$

When $H_{ind}(V-F, H)$ is connected, all units in $V-F$ can be diagnosed correctly to be fault-free. Contrarily, when $H_{ind}(V-F, H)$ is disconnected, according to Lemma 11, $C_{comp}(H_{ind}(V-F)) = \{C_1, C_2, C_3, C_4\}$ ($V(C_3)$, $V(C_4)$ may be empty). Now three situations described as follows will be considered.

Case 14.1 $|V(C_1)| = 2^n - 3 - |F|$, $|V(C_2)| = 1$, $|V(C_3)| = 1$ and $|V(C_4)| = 1$.

Similarly, $|V(C_1)| \geqslant |F| + 1$, then the system identifies all units in $V(C_1)$ correctly as fault-free. It is claimed that each unit of F is connected to $V(C_1)$, otherwise $\sum_{k=1}^{3} k\,|Card_k(H_{ind}(V-F))| \geqslant 4$ which is a contradiction to the hypothesis. Therefore, the system can diagnose all units of F to be faulty through units in $V(C_1)$.

Case 14.2 $|V(C_1)| = 2^n - 3 - |F|$, $|V(C_2)| = 2$, $|V(C_3)| = 1$ and $|V(C_4)| = 0$.

Similarly, $V(C_1)$ can be identified correctly as fault-free. Similar to case 14.1, it is claimed that each unit of F is connected to $V(C_1)$. Therefore, all units of F can be identified as faulty by fault-free set $V(C_1)$.

Case 14.3 $|V(C_1)| = 2^n - 2 - |F|$, $|V(C_2)|$

$= 2$, $|V(C_3)| = 0$ and $|V(C_4)| = 0$.

Similarly, $V(C_1)$ can be identified correctly as fault-free. According to Lemma 11, there exists at most one unit in F, say u, which is disconnected to any one unit in $V(C_1)$. Therefore, the system can diagnose correctly all units in $F - \{u\}$ to be faulty by the units in $V(C_1)$. It is easily seen that the test results of C_2 are 0 and the test results from $V(C_2)$ to u are 1. Then $V(C_2)$ can be identified as fault-free.

Case 14.4 $|V(C_1)| = 2^n - 2 - |F|$, $|V(C_2)| = 1$, $|V(C_3)| = 1$ and $|V(C_4)| = 0$.

Similarly, $V(C_1)$ can be identified correctly as fault-free. And each unit of F is connected to $V(C_1)$, otherwise $|F| > 3n - 5$. Therefore, the system can diagnose correctly all units in $F - \{u\}$ to be faulty by the units in $V(C_1)$.

3.2 The diagnosability of $t/(t+1)$- diagnosable of 2D(3D)-mesh

Lemma 12 Let $H(V, E)$ denote a 2D-mesh with n^2 units ($n \geqslant 5$), $S = \{u, v, w\} \subset V$ be a set in which u, v, w have a common neighbor. Then $|N(S)| = 8$.

Proof According to the symmetry of the graph, S can be shown in Fig. 5. It is obviously that $|N(S)| = 8$.

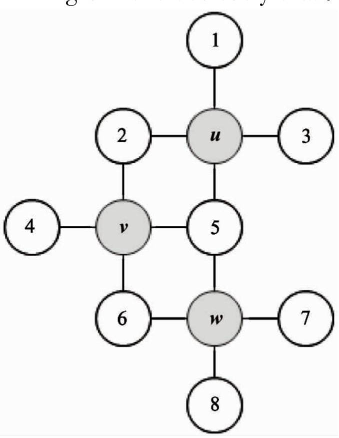

Fig. 5 A subset $S = \{u, v, w\}$ of Lemma 12 in 2D-mesh

Theorem 7 A 2D-mesh with n^2 units ($n \geqslant 5$) is not 9/10-diagnosable.

Proof According to Lemma 3 and Lemma 12, it is easily seen that the 2D-mesh with n^2 units ($n \geqslant 5$) is not 9/10-diagnosable.

According to the structure of 2D-mesh with n^2 units ($n \geqslant 5$), it is easy to get the following Lemmas.

Lemma 13 For a 2D-mesh with n^2 units ($n \geqslant 5$), denoted by $H(V, E)$, let $S \subset V$ with $|S| = 6$. If $H_{ind}(V - S)$ is disconnected, then:

i) Either $C_{comp}(H_{ind}(V - S)) = \{C_1, C_2\}$ with $|V(C_1)| = 1$ and $|V(C_2)| = n^2 - 1 - |S|$.

ii) Or $C_{comp}(H_{ind}(V - S)) = \{C_1, C_2\}$ with $|V(C_1)| = 2$ and $|V(C_2)| = n^2 - 2 - |S|$.

Lemma 14 For a 2D-mesh with n^2 units ($n \geqslant 5$), denoted by $H(V, E)$, let $S \subset V$ with $|S| = 7$. If $H_{ind}(V - S)$ is disconnected, then:

i) Either $C_{comp}(H_{ind}(V - S)) = \{C_1, C_2\}$ with $|V(C_1)| = 1$ or 2 or 3 and $|V(C_2)| = n^2 - |V(C_1)| - |S|$.

ii) Or $C_{comp}(H_{ind}(V - S)) = \{C_1, C_2, C_3\}$ with $|V(C_1)| = 1$, $|V(C_2)| = 1$ and $|V(C_3)| = n^2 - 2 - |S|$.

Lemma 15 For a 2D-mesh with n^2 units ($n \geqslant 5$), denoted by $H(V, E)$, let $S \subset V$ with $|S| = 8$. If $H_{ind}(V - S)$ is disconnected, then:

i) Either $C_{comp}(H_{ind}(V - S)) = \{C_1, C_2\}$ with $|V(C_1)| = 1$ or 2 or 3 or 4 and $|V(C_2)| = n^2 - |V(C_1)| - |S|$.

ii) Or $C_{comp}(H_{ind}(V - S)) = \{C_1, C_2, C_3\}$ with $|V(C_1)| = 1$, $|V(C_2)| = 1$ and $|V(C_3)| = n^2 - 2 - |S|$.

iii) Or $C_{comp}(H_{ind}(V - S)) = \{C_1, C_2, C_3, C_4\}$ with $|V(C_1)| = 1$, $|V(C_2)| = 1$, $|V(C_3)| = 1$ and $|V(C_4)| = n^2 - 3 - |S|$.

Theorem 8 A 2D-mesh with units ($n \geqslant 5$) is 8/9-diagnosable.

Proof A similar argument of Theorem 6 can be used.

Similar to 2D-mesh, a 2D-mesh with n^3 units is 11/12- diagnosable but not 2/13-diagnosable.

4 Comparisons between $t/(t+1)$-diagnosability and other classical diagnosabilities

In this section, a comparison is taken between the $t/(t+1)$-diagnosability and the t-diagnosability (t/t-diagnosability) in an n-dimensional hypercube and 2D (3D)-mesh.

According to Theorem 6, the $t/(t+1)$-diagnosability of Q_n ($n \geqslant 5$) is $3n - 5$. On the other hand, it's known that the t-diagnosability[22] (t/t-diagnosability[19]) of Q_n is $n(2n-2)$. For the sake of convenience, a table is set up (see Table1 for the comparison of above three diagnosabilities of Q_n ($n \geqslant 5$). Besides, by Theorem 8, a table (see Table 2) can be made for the comparison between two of $t/(t+1)$-diagnosability, t-diagnosability and t/t-diagnosability of 2D-mesh (3D-mesh). According to Table 1 and Table 2, one can see the $t/(t+1)$-diagnosability is much larger than the classical diagnosability. Especially, for an n-dimensional hypercube, the $t/(t+1)$-diagnosability is about 2 times as large as t-diagnosability and is also about 1.5 times as large as t/t-diagnosability.

Table 1 The comparison between the $t/(t+1)$-diagnosability and the t-diagnosability (t/t-diagnosability) in Q_n

Dimension	5	6	7	8	9	10
t-diagnosability	5	6	7	8	9	10
t/t-diagnosability	8	10	12	14	16	18
$t/(t+1)$-diagnosability	10	13	16	19	22	25

Table 2 The comparison between the $t/(t+1)$-diagnosability and the t-diagnosability (t/t-diagnosability) in 2D (3D)-mesh

	2D-mesh	3D-mesh
t-diagnosability	4	6
t/t-diagnosability	6	10
$t/(t+1)$-diagnosability	8	11

5 Conclusions

In this paper, the concept of t/t-diagnosable system is extended to a novel concept, called $t/(t+1)$-diagnosable system, the latter has larger diagnosability than the former. A necessary and sufficient condition is presented to test whether a system is $t/(t+1)$-diagnosable. At the same time, in order to figure out the diagnosability of a system, some significant properties of $t/(t+1)$-diagnosable system are proposed. Furthermore, it shows an n-dimension hypercube ($n \geqslant 5$) is $(3n-5)/(3n-4)$-diagnosable, and a $2D$-mesh ($3D$-mesh) with $n^2(n^3)$ units is 8/9-diagnosable(11/12-diagnosable) of $Q_n(n \geqslant 5)$ is about 3 times as large as its t-diagnosability and is 1.5 times as large as its t/t-diagnosability.

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Xie Min, born in 1974. She is currently pursuing the Ph. D degree from the Institute of Automation, South China University of Technology. She received her B. S. and M. S. degrees in water resources and hydroelectric power engineering from North China University of Water Resources and Electric Power, China, in 1997 and 2000 respectively. And she is currently an associate professor in the School of Computer, Electronics and Information, Guangxi University, China. Her research interests include network fault tolerance, interconnection networks, fault diagnosis and algorithm theory.